Stability of High Froude Number Flows along an Incline

Thesis

Submitted in Partial Fulfillment of the requirements of

BITS C421T/422T Thesis

By

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10TH MAY, 2010.

ACKNOWLEDGEMENT

I would like to thank Dr. Rama Govindarajan, Chair EMU, JNCASR, for providing me this opportunity to work in the exciting area of interfacial instabilities in fluid flow down an incline. This work would not have been possible if not for Dr. Rama Govindarajan's constant encouragement and guidance. I would also like to thank Prof. Ravi Prakash, Dean RCD, and Prof. Gunjan Soni for encouraging me to carry out this off-campus thesis. I would like to thank Ratul Dasgupta, Vishwanath Shastry and Dr. Rama Govindarajan for their help and guidance with the code for solving the weakly non-similar equations numerically. Finally, I would like to express my gratitude to all students of the Engineering Mechanical Unit, JNCASR, for the numerous group discussions without which a clear understanding of the theoretical background needed for this work.

CERTIFICATE

This is to certify that the Thesis entitled, "***Stability of High Froude Number Flows along an Incline.***" and submitted by **Sarath Ramadurgam** ID No. **2005A3B5134P** in partial fulfillment of the requirement of BITS C421T/422T Thesis embodies the work done by him under my supervision.

Date: Signature of the Supervisor

Name

Designation

List of Symbols and Abbreviations

φ	Stream Function
Re	Reynolds Number
Fr	Froude Number
S	Weber Number
g	Acceleration due to gravity
ρ	Density
μ	Viscosity
ν	Kinematic Viscosity
T	Surface Tension
u_a	Average Velocity
U	Non-dimensional velocity profile
h	Fluid height
Q	Flow Rate
β	Angle of Inclination
η	Non-dimensionalised stream wise coordinate
ξ	Non-dimensionalised normal coordinate
α	Wave number
c	Wave velocity
λ	Wave length

ABSTRACT

Thesis Title: Stability of High Froude Number Flows along an Incline.

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Semester: Second Session: 2009-10

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Abstract:

Half-Poiseuille flow is a solution of fluid flow down planar inclines obtained only for specific values of inlet Flow Rate (Q) and Froude Number (Fr). In the current work, the effect of inlet conditions on base flow and its stability has been studied. For an inlet with a 'high Froude Number', flows may accelerate and become thinner or decelerate and experience a hydraulic jump along the incline. Base flow for various inlet conditions and inclinations has been obtained by numerically solving the weakly non-similar equations. The dependence of the base flow on the inlet flow rate and Froude number has been presented in detail. The analysis here has been restricted to small inclination and small Reynolds Number flows where linear stability analysis is accurate. The linear stability of the base flow for various inlet conditions has been studied and neutral stability boundary has been compared to the expression obtained in for parabolic parallel flow down an incline by Yih, 1963.

Table of Contents

1. Introduction

Film flows down an incline exhibit a plethora of complex dynamics which are of relevance in geophysical situations as well as in engineering processes [16, 11]. Studies on the stability of laminar film flows comprise too large a volume of literature to recount here, but a few most relevant are mentioned. The earliest analytical studies of the linear stability of thin film flows were carried out by Benjamin [1] and Yih [2] where the Orr-Sommerfeld equation was solved analytically for long-wavelength perturbations and small Reynolds number flows. The linear stability work was later extended to higher Reynolds number flows [5] and the most unstable modes were obtained numerically and showed that there were insignificant differences between the temporal and spatial methods of evolution for Reynolds numbers up to 1000. In [6], stability of non-parabolic bases flow with a form factor studied. In these studies, the resulting equation may be posed as a generalized linear eigenvalue problem. At finite wavelengths however, the boundary conditions result in a quadratic eigenvalue problem, which has been solved by [9].

A nonlinear approach was first employed by Benney [3] for isothermal laminar flow on an inclined plane, and he derived a nonlinear equation of evolution for the film interface. Benney was credited for using the perturbation method to derive the first simplified model for describing the instantaneous amplitude of surface waves on liquid films. The model is commonly known as the long wave LW equation in the literature. Fully nonlinear treatment for film thickness and its stability are available in literature [3, 8, 13 and 15] which are not important for the current work since we focus on flows down small inclinations where 2^{nd} and higher order variations in fluid height along the stream-wise directions is negligible. Hence the effects of surface tension do not appear in the equations while solving for the base flow.

The present work differs from the earlier stability studies that we know of in a significant respect: we study the region where the base flow evolves with downstream distance x. Here the film height and therefore the Froude number are functions of x, and the velocity profile has not attained its parabolic state. Most of the earlier works on the linear stability of thin film flows have assumed parallel flow with half-Poiseuille velocity profile [1, 2 5].

Consider a parallel flow down an inclination β, height h and $u_a = \frac{gh^2 \sin\beta}{3\nu}$. [2]

$$Re = \frac{u_a h}{\nu} \qquad -(1)$$

$$Fr_{parallel} = \frac{u_a}{(g\cos\beta\, h)^{1/2}} \qquad -(2)$$

The equation for u_a can be written in terms of Re and Fr as:

$$Fr_{parallel}^2 = \frac{Re\tan\beta}{3} \qquad -(3)$$

In terms of flow rate Q and height h:

$$Q = \frac{gh^3 \sin\beta}{3\nu} \qquad -(4)$$

Thus, clearly (3) is the mass balance for a half-Poiseuille flow down an incline of β. For a given Q if the inlet height is reduced i.e. the inlet Froude number is increased beyond $Fr_{parallel}$ then half-Poiseuille parallel flow can no more be a solution. To the best of our knowledge, there has been no study on such “high Froude number” flows down an incline where the flow becomes non-parabolic and the local Froude number varies in the stream-wise direction. The case of inlet Froude lower than $Fr_{parallel}$ has also been studied. In the next section, the case of 2-D parallel flow and its stability and the numerical scheme used to solve the Orr-Sommerfeld Equation has been discussed followed by the formulation of the weakly non-similar equation has been presented. In the subsequent sections numerical solutions and the linear stability of such flows have been presented.

2. 2-D Parallel Flow

2.1 Orr-Sommerfeld Equation

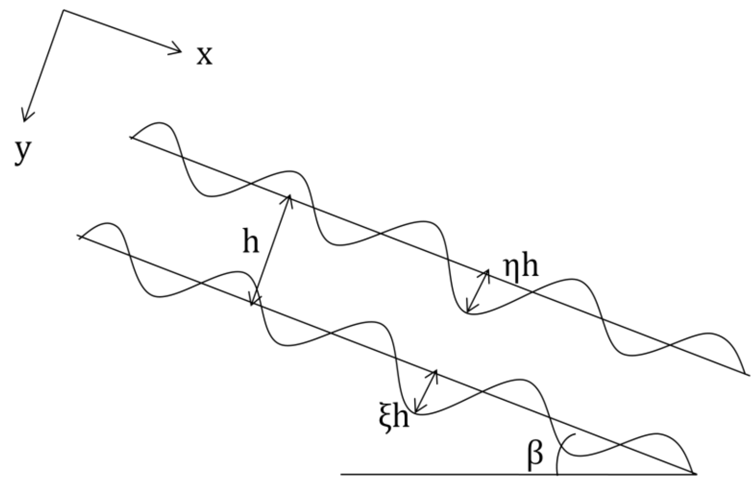

Fig. 1: Thin film flow along an incline of inclination β.

Here, h is the unperturbed film thickness; ηh is the perturbation amplitude of the free surface; ξh is the perturbation amplitude of the solid surface (which is *zero* for the current work); U is the mean velocity along x axis; u and v are the velocity perturbations along x and y axis respectively.

Using the following definitions to rescale the Navier-Stokes equation and obtain all expressions in the non-dimensional form.

$$x = hx', \quad y = hy', \quad p = \frac{p'}{\rho u_a^2}, \quad t = \frac{t' u_a}{h}, \quad u_a = \frac{gh^2 \sin\beta}{2\nu}, \quad Re = \frac{u_a h}{\nu}, \quad Fr = \frac{u_a}{(gd)^{1/2}}$$

Here, Re is the Reynolds number, Fr is Froude number, u_a is the average velocity of the primary flow.

Thus solving the laminar flow with the above non-dimensional variables the mean flow is given as:

$$U(y) = (1 - y^2) \quad - (5)$$

Here, y has been rescaled from (0, h) to (0, 1), where y is 0 at the unperturbed water surface.

The two-dimensional linear perturbation dynamics of the film flow is governed by the Navier-Stokes equations linearized around the basic state (5) derived above [1, 2]. In the current case considered, the solid incline is assumed to be flat i.e. ξ = 0.

The perturbation in pressure ($P = P_0 + p$) is taken as p and Δ is the Laplacian operator.

The linearized Navier-Stokes equations and continuity equation are:

$$u_t + Uu_x + U_y v = -p_x + \left(\frac{1}{Re}\right)\Delta u \quad -(6)$$

$$v_t + Uv_x = -p_y + \left(\frac{1}{Re}\right)\Delta v \quad -(7)$$

$$u_x + v_y = 0 \qquad -(8)$$

From equation (8) we can write the perturbed velocities u and v in terms of a dimensionless stream function ψ. $u = \psi_y, \; v = -\psi_x$.

The no slip and stress free boundary conditions can be written as:

$$\psi_y = 0, \qquad -\psi_x = 0 \quad at\ y = 1 \qquad -(i)$$

$$U_{yy}\eta + \psi_{yy} - \psi_{xx} = 0 \;\; at\ y = 0 \qquad -(ii)$$

$$\left(\frac{\cos\beta}{Fr^2}\right)\eta + p + \left(\frac{1}{Re}\right)\psi_{xy} - \mathrm{S}\eta_{\mathrm{xx}} = 0\ at\ y = 0 - (\mathrm{iii})$$

Here $\mathrm{S} = \frac{\mathrm{T}}{\rho \mathrm{h} \mathrm{u}_{\mathrm{a}}^2}$ where T is the surface tension.

The Kinematic condition at the free surface is:

$$-\psi_x = \eta_t + U\eta_x \quad at\ y = 0 \qquad -(iv)$$

Since the film is thin, the scaled x axis is (-inf, +inf). Thus a sinusoidal disturbance is assumed.

$$\psi = \varphi(y)e^{i\alpha(x-ct)}, \quad p = f(y)e^{i\alpha(x-ct)} \quad -(9)$$

Using (9) in equations (6-8) results in the Orr-Sommerfeld equation ($\varphi' = \frac{d\varphi}{dy}$):

$$\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha Re[(U-c)(\varphi'' - \alpha^2\varphi) - U''\varphi] \quad -(10)$$

From equation (9) and above boundary conditions (i-iv) now read as:

$$\varphi'(1) = 0 \qquad -(11)$$

$$\varphi(1) = 0 \qquad -(12)$$

$$\varphi''(0) + \left(\alpha^2 - \frac{2}{c-1}\right)\varphi(0) \qquad -(13)$$

$$[\alpha(3\cot\beta + \alpha^2 SRe)]\frac{\varphi(0)}{c-1} + \alpha(\mathrm{Re}(c-1) + 3\alpha i)\varphi'(0) - i\varphi'''(0) = 0 \quad -(14)$$

Equations (10-14) constitute a nonlinear eigenvalue problem. Benjamin and Yih proved that there definitely exist unstable modes given the following condition:

$$Re > \frac{5}{6}\cot\beta \qquad -(15)$$

2.2 Numerical Scheme

Using the Spectral methods and MATLAB [10] linear eigenvalue problems can be easily solved. But, the eigenvalue problem derived above is nonlinear due to equation (14). Thus we convert it into an iterative linear eigenvalue problem:

$$[\alpha(3\cot\beta + \alpha^2 SRe)]\frac{\varphi(0)}{c_g-1} + \alpha(\mathrm{Re}(c-1) + 3\alpha i)\varphi'(0) - i\varphi'''(0) = 0 \quad -(16)$$

Here, c_g is the value of c (the fastest growing unstable mode within physical limits is taken) calculated in the previous iteration. For the first run and small α, c_g =2.

Thus equations (10-13 and 15) constitute a linear eigenvalue problem for a given c_g. Using Chebyshev Differential matrices and *eig* routine of MATLAB, the eigenvalue problem is solved. The results have been compared with ones from [9] and have been found to match accurately.

3. Weakly Non-Similar Equations

3.1 Formulation of the problem

Let us consider a two dimensional flow along a flat plate held at an inclination of β with respect to the horizontal. The continuity and momentum equations that govern the steady base flow are:

$$u_x + v_y = 0 \qquad -(17)$$

$$uu_x + vu_y = -\frac{1}{\rho}p_x + \nu(u_{xx} + u_{yy}) + g\cos\beta \qquad -(18)$$

$$uv_x + vv_y = -\frac{1}{\rho}p_y + \nu(v_{xx} + v_{yy}) - g\sin\beta \qquad -(19)$$

$$Q = \int_0^{h(x)} u(y)dy \qquad -(20)$$

Here u and v are the velocities along the stream-wise direction x and wall normal direction y respectively, p is the pressure, g is the acceleration due to gravity, Q is the flow rate which is kept constant and h(x) is the height of the liquid film.

The flow is assumed to be weakly non-parallel i.e. dh/dx is small, 2nd and higher order variations are neglected. Hence the surface tension effects are not important for finding the base-flow and the hydrostatic assumption is valid. Equations (18-19) reduce to:

$$uu_x + vu_y = -gh'\cos\beta + \nu(u_{xx} + u_{yy}) + g\sin\beta \qquad -(21)$$

Here, $h' = dh/dx$. The stream-wise coordinate x is non-dimensionalised using $dx = h(x)d\xi$, normal coordinate y is non-dimensionalised using $y = h(x)\eta$, and the stream function is assumed to have the from $\varphi = Qf(\eta, \xi)$. The velocities can be written in terms of the stream function as:

$$u = \frac{Q}{h} f_\eta \qquad - (22)$$

$$v = \frac{Q}{h} (\eta h' f_\eta - f_\xi) \qquad - (23)$$

Assuming locally self-similar flow, on substituting (22) and (23) into (21) and retaining terms only up to the first-derivative in ξ we arrive at the weakly non-similar equation for flow along an incline similar to the expression obtained in [17]:

$$\boldsymbol{f_{\eta\eta\eta} - \frac{Re}{Fr^2}(h' - \tan\beta) + Re h' f_\eta^2 = Re\left(f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\right)} \qquad - (\mathbf{24})$$

Here the local Froude number is $Fr = \frac{Q}{(g \cos\beta h^3)^{1/2}}$ and the Reynolds number, which is a constant since the flow rate is kept constant, is $Re = \frac{Q}{\nu}$. For flow over a horizontal the term $\{\frac{Re}{Fr^2}(\tan\beta)\}$ becomes 0 thus the expression reduces to the one obtained in [17].

Substituting (22) and (23) into the no slip boundary condition and tangential stress balance condition to the first order of h' yields:

$$f(0, \xi) = 0 \qquad - (25)$$

$$f_\eta(0, \xi) = 0 \qquad - (26)$$

$$f_{\eta\eta}(1, \xi) = 0 \qquad - (27)$$

Here h' is an unknown quantity, hence for a given Fr it is obtained from mass flow conservation:

$$f(1, \xi) = 0 \qquad - (28)$$

Thus to obtain the base-flow, (24) and (19) are numerically solved subject to boundary conditions (25-27). From the numerical solutions presented in the next section, it is observed that $Reh' \cong 1{:}81$ at high Froude numbers for any angle of inclination. Thus for the weakly non-parallel flow assumption to hold good, Reynolds number has to be large. For the numerical solution of (24); $\xi \equiv \nu\xi$ (i.e. $x \equiv \nu x$) and $\tan\beta \equiv \frac{tan\,\beta}{\nu}$. This implies that the numerical solutions to (24) are restricted to either very large Reynolds numbers or extremely small inclinations unless ν is very large.

Here on all values of x and β as shown in the plots are related to the actual physical values as follows:

$$x \equiv \nu x_{physical}$$

$$\tan\beta = \frac{tan\,\beta_{physical}}{\nu}$$

As a first approximation, the solutions obtained can be treated to be valid for small Reynolds numbers and large β, i.e. large ν, even though terms involving h'', h'^2, h'^3, $h'f_\xi$, etc. are no more negligible.

3.2 Numerical Results

Our analysis has been limited to $Fr \equiv (0.2 - 100)$. For a given value of β, (24) is an ordinary differential equation in η. The numerical solution of the base flow is obtained in the exact same way as described in [17]. On Solving (24), depending on the inclination and flow rate, generally two branches, referred to from here on as *branch 1* and *branch 2*, of solutions are obtained. The *branch 1* solutions may go through a hydraulic jump depending on the value of flow rate and angle of inclination as shown in Figure 2. Some typical velocity profiles are presented in Figure 7.

One condition for *branch 1* solutions to go through a hydraulic jump can be guessed to be:

$$Fr_{parallel} < 1 \qquad -(29)$$

Clearly if the above condition is not satisfied, the flow becomes parallel before going through a Froude of 1, i.e. doesn't experience a jump. From the *branch 1* solutions obtained for various angles of inclination as shown in Figure 3, one more condition can be inferred for the occurrence of a hydraulic jump, which is:

$$h' - tan\,\beta > 0 \qquad - (30)$$

To understand this condition analytically, in (8), consider high Froude number flow where $Reh' \cong 1{:}81$, hence $\frac{Re}{Fr^2}(h' - \tan\beta)$ is constant and the variation of f with respect to ξ, and h' are neglected. (24) becomes:

$$f_{\eta\eta\eta} - \frac{Re}{Fr^2}(h' - \tan\beta) = 0$$

Solving the equation and applying the boundary conditions yields:

$$u(\eta) = -\frac{gh^2 \sin\beta}{2\nu}(h' - \tan\beta)(2\eta - \eta^2) \qquad - (31)$$

Clearly, the half-Poiseuille solution is got back if h' = 0. Here since h' is a constant, the sign of $(h' - \tan\beta)$ completely alters the form of velocity profile. From the solutions obtained for higher angles of inclination with Q kept as a constant, as shown in Figure 3 it is clear that when the term $(h' - tan\,\beta > 0)$, *branch 1* solutions go through a hydraulic jump. Thus, when angle of inclination is such that $Re\,tan\,\beta \geq 1.81$ *branch 1* solutions do not go through a hydraulic jump. The *branch 1* solutions that do go through a jump can be obtained for the entire Froude number range and h'/ν takes a positive value. These flows undergo separation post jump. If the inlet conditions are such that $(h' - tan\,\beta < 0)$, then as shown in the figure, *branch 1* solutions do not go through a jump. For this set of initial conditions, the numerical solution seems to fail as local Froude number approaches the $Fr_{parallel}$. For the same initial conditions, another set of solutions, branch 2, are obtained for a Froude number range (0.2-0.65) as shows in Figure 4. For this branch, h'/ν goes through a zero at $Fr_{parallel}$. In this branch of solutions the velocity profiles closely resemble half-Poiseuille flows. Figure 5 shows that for a fixed inclination, branch 1 solutions go through a hydraulic jump only for those values of Q such that $Re\tan\beta < Re\,h'$.

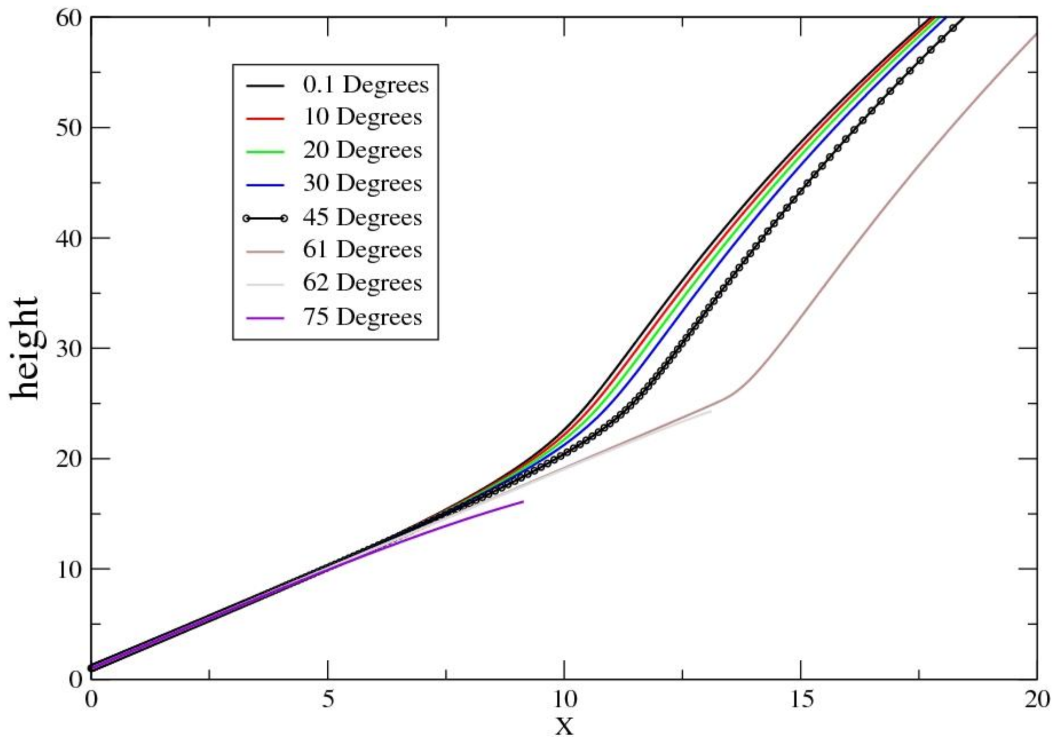

Fig. 2: Height versus x for Q=1 and various inclinations. (*branch 1*)

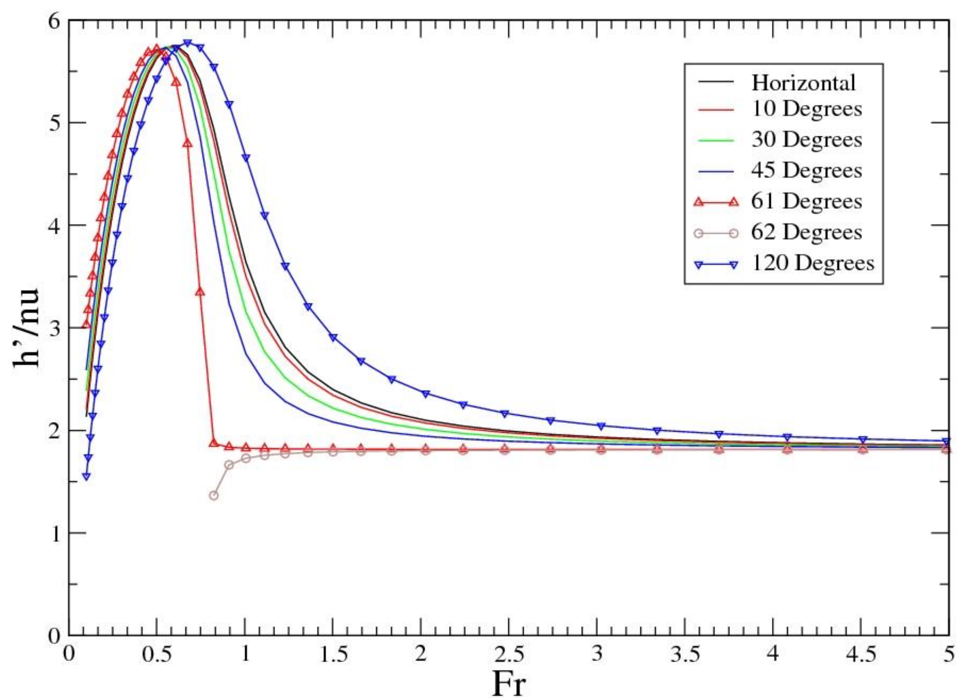

Fig. 3: Reh' versus Fr for Q=1 and various inclination. (*branch 1*)

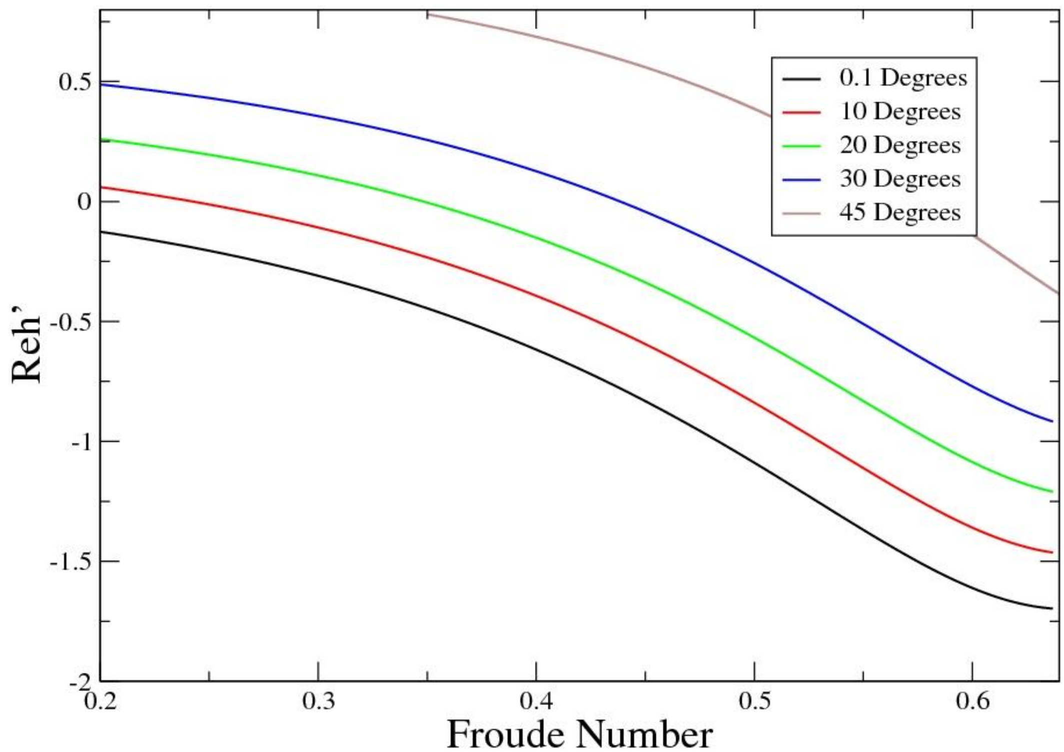

Fig. 4: Reh' versus Fr for Q=1 and various inclination. (*branch 2*)

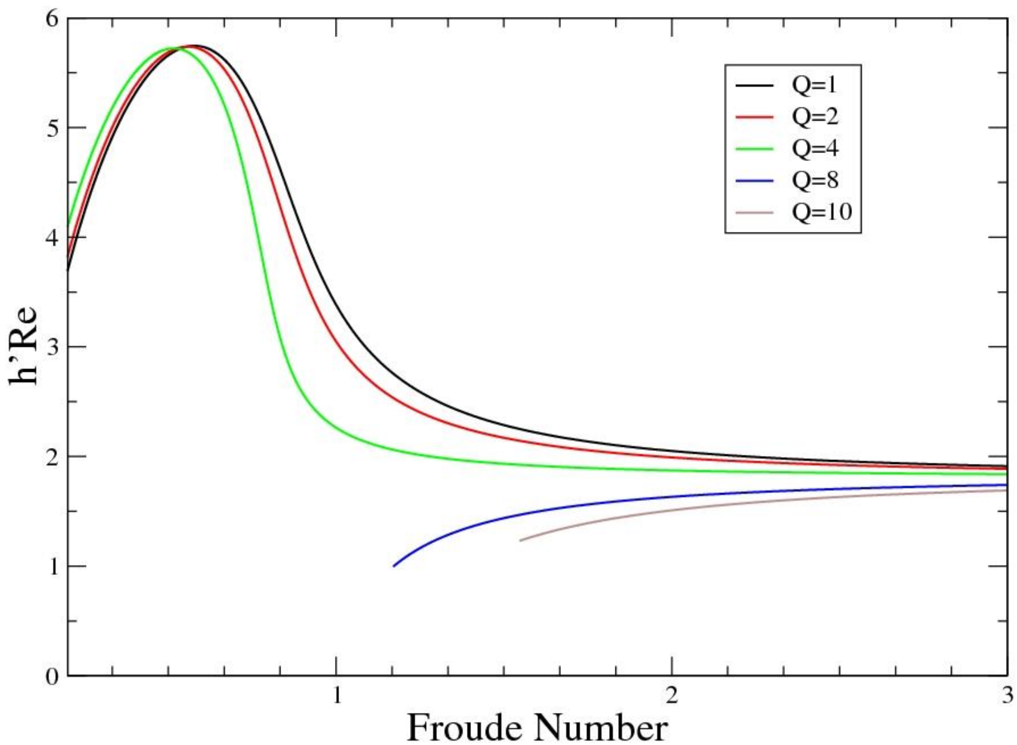

Fig. 5: Reh' versus Fr for β=20 and various flow rates. (*branch 1*)

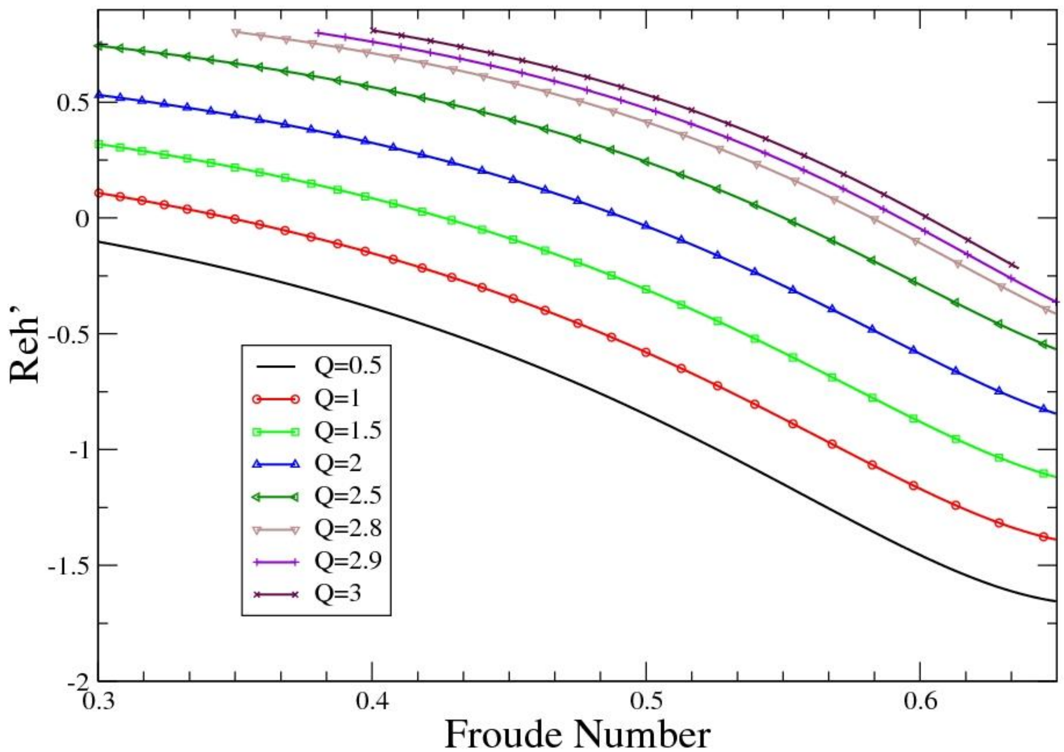

Fig. 6: Reh' versus Fr for β=20 and various flow rates. (*branch 2*)

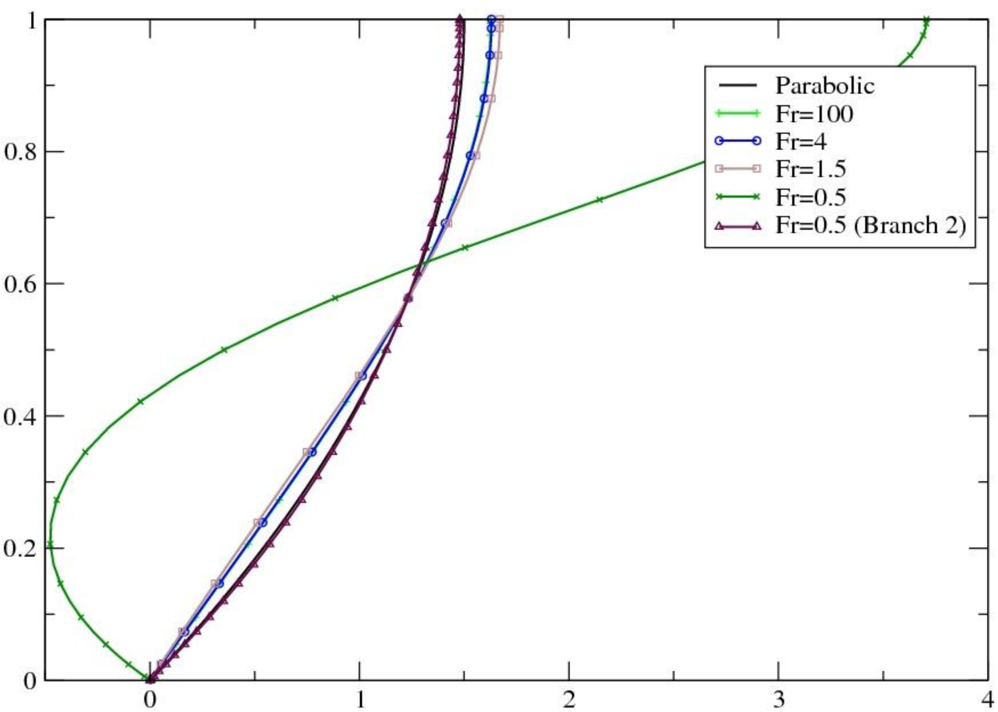

Fig. 7: y versus u for β=20 and various Fr. (*branch 1, 2 and Parabolic*)

4. Linear Stability Analysis

The derivation of the Orr-Sommerfeld Equation has been presented in Section 2.1. Here a more generalized set of equations application for any base flow have been solved. They are:

$$\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha Re[(U-c)(\varphi'' - \alpha^2\varphi) - U''\varphi] \quad -(32)$$

$$\varphi'(0) = 0 \qquad -(33)$$

$$\varphi(0) = 0 \qquad -(34)$$

$$\varphi''(1) + \left(\alpha^2 - \frac{U''(1)}{c - U(1)}\right)\varphi(1) \qquad -(35)$$

$$\left[\frac{\alpha Re}{Fr^2} + \alpha^3 SRe\right]\frac{\varphi(1)}{c - U(1)} + \alpha \mathrm{Re}\mathrm{U}'(1)\varphi(1) + \alpha(\mathrm{Re}(\mathrm{c} - \mathrm{U}(1)) + 3\alpha\mathrm{i})\varphi'(1) - i\varphi'''(1) = 0 \qquad -(36)$$

Though the numerical results obtained in the previous section are applicable for very small angles of inclination, as a first approximation we study the stability of profiles assuming they are valid for low Reynolds number case. In [2], a relation between the critical Reynolds number and angle of inclination has been obtained, which is:

$$Re_{crit} > \frac{5}{6}\cot\beta$$

A flow along a 20^{o} inclination is considered. In Figure 8 it is clear that for a Froude of 0.500 and flow rate Q = 2:65 the flow just begins to get unstable. It is unstable for Q > 2:65. But, going by the relation described in [2], the critical Reynolds number should be 2:2896. Clearly, this quantity Q = 2:65 is dependent on inclination as well as Froude Number shown in Figure 9. From Yih's relation the Froude Number corresponding to Re_{crit} is $Fr = 0.5270$ as shown in Figure 9 as a plus mark.

Figures 10-14 are show the effect of inclination and flow rate on stability of the velocity profiles numerically obtained by solving the weakly non-similar equations.

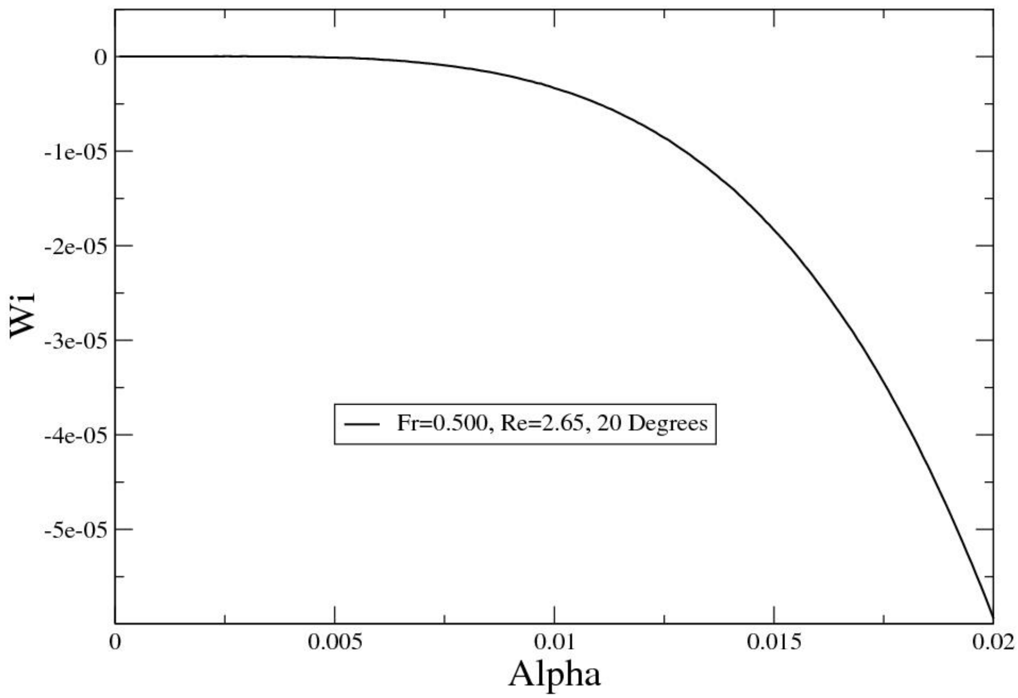

Fig. 8: Wi_{max} versus α for Fr = 0.5000, Q = 2.65 and β=20. (*branch 2*)

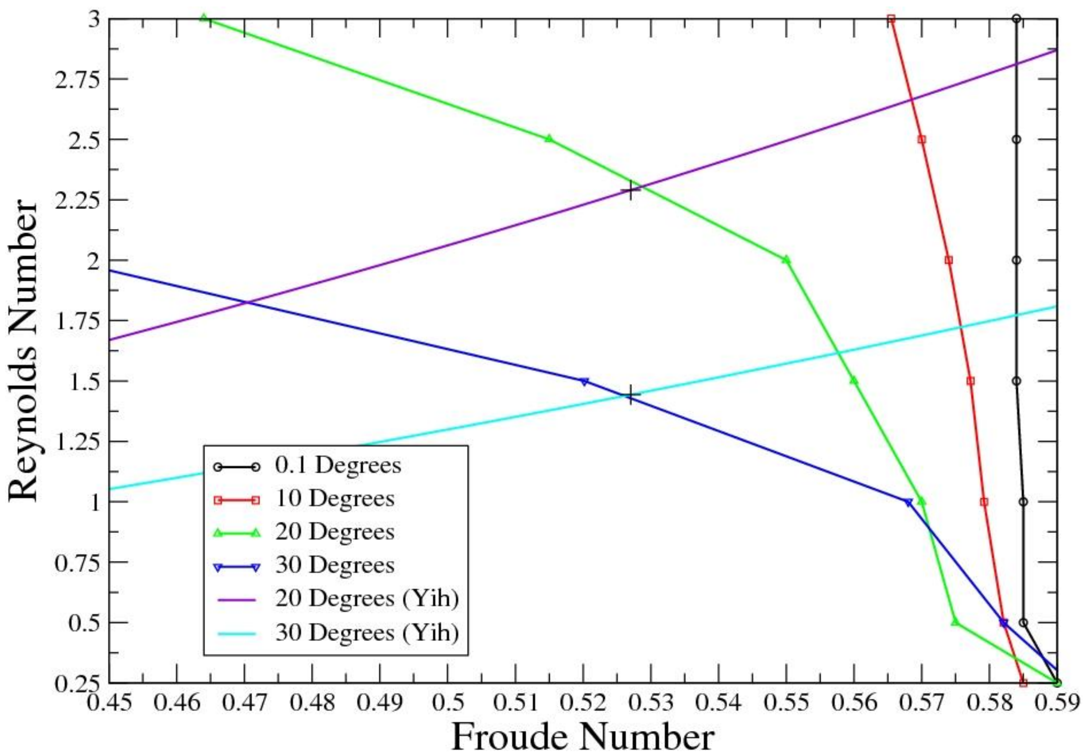

Fig. 9: Re versus Fr showing the neutral stability boundary for various β. (*branch 2*)

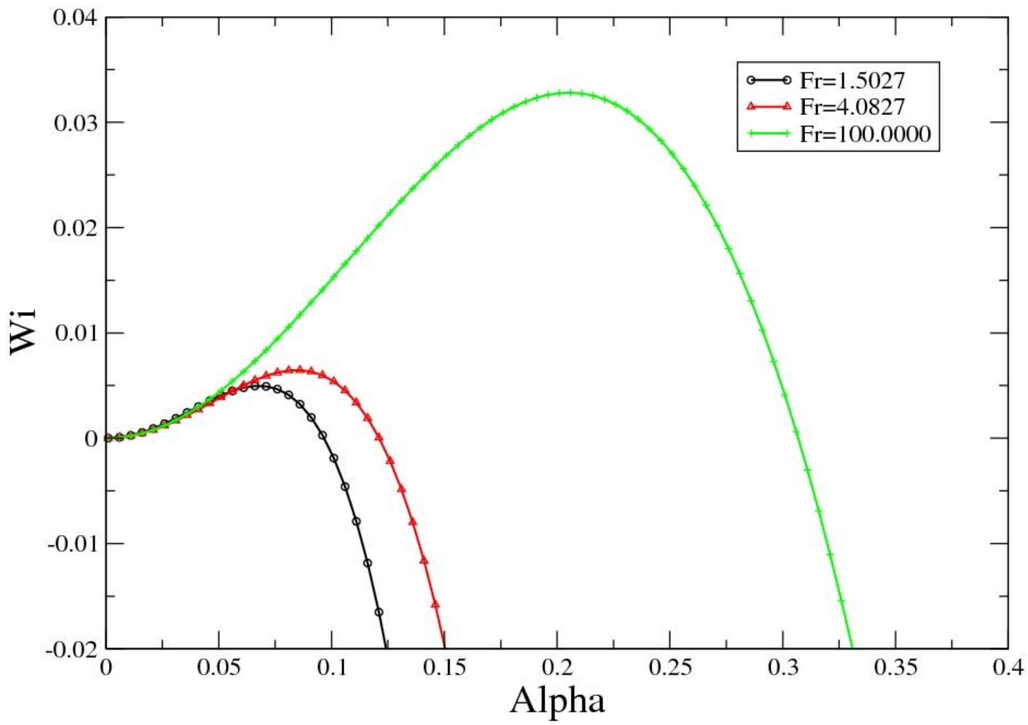

Fig. 10: Wi_{max} versus α for Q = 1, β=20 and various Fr. (*branch 1*)

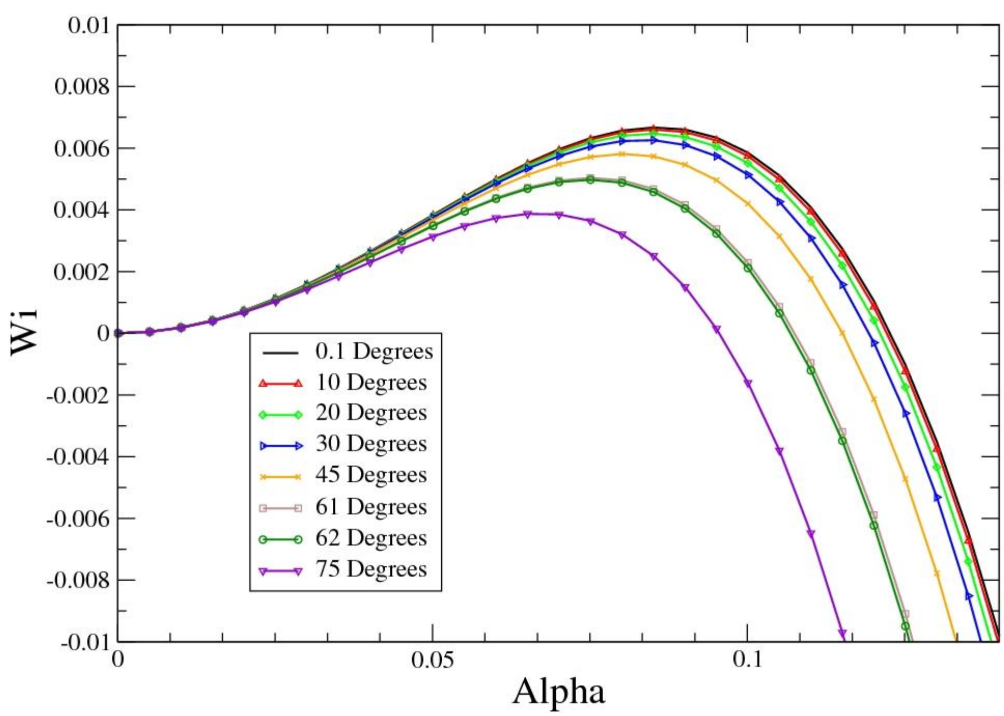

Fig. 11: Wi_{max} versus α for Q = 1, Fr=4.0827 and various β. (*branch 1*)

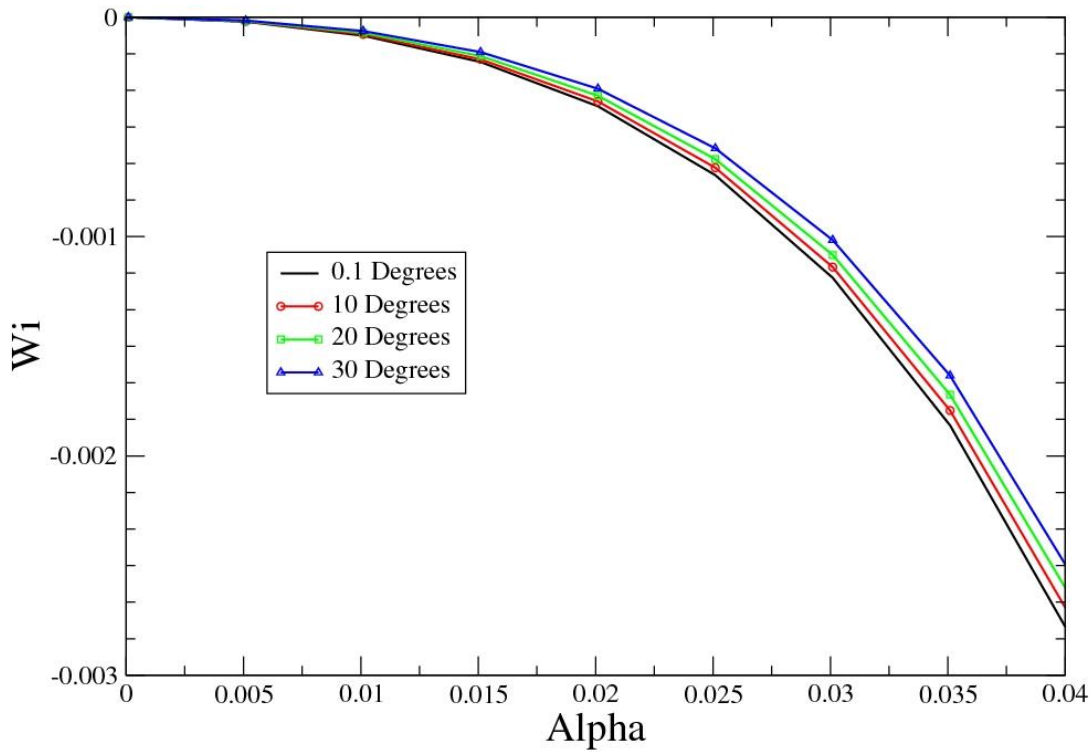

Fig. 12: Wi_{max} versus α for Q = 1, Fr=0.4000 and various β. (*branch 2*)

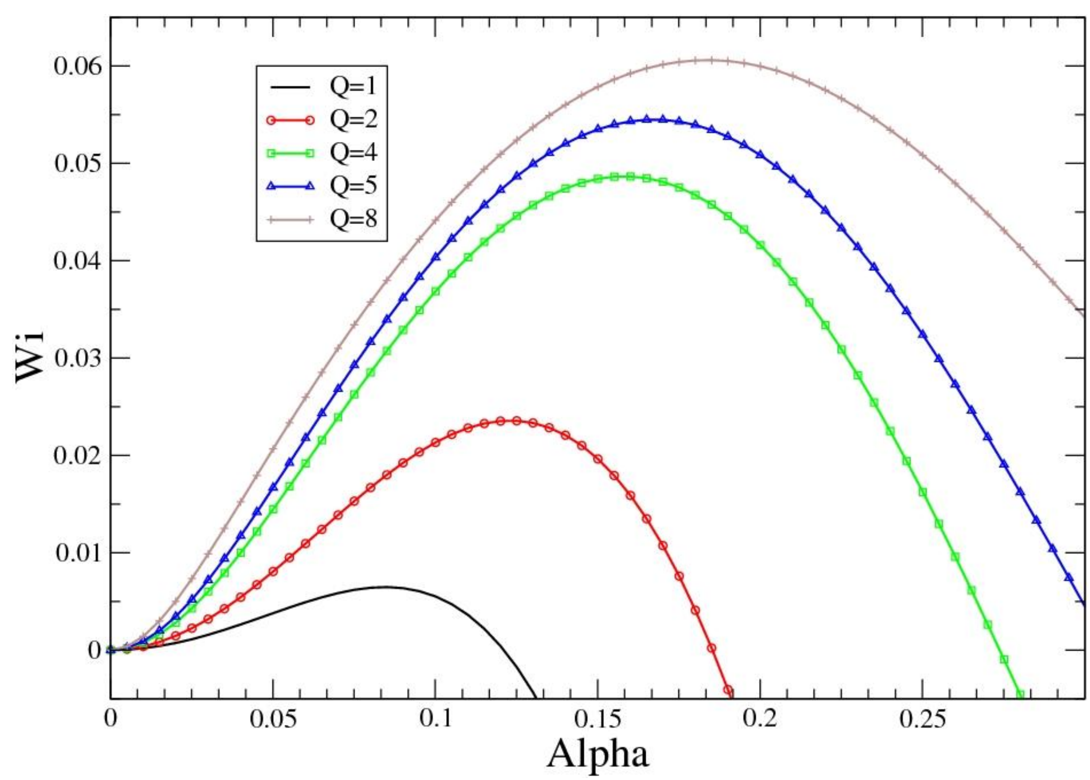

Fig. 13: Wi_{max} versus α for β = 20, Fr=4.0827 and various Q. (*branch 1*)

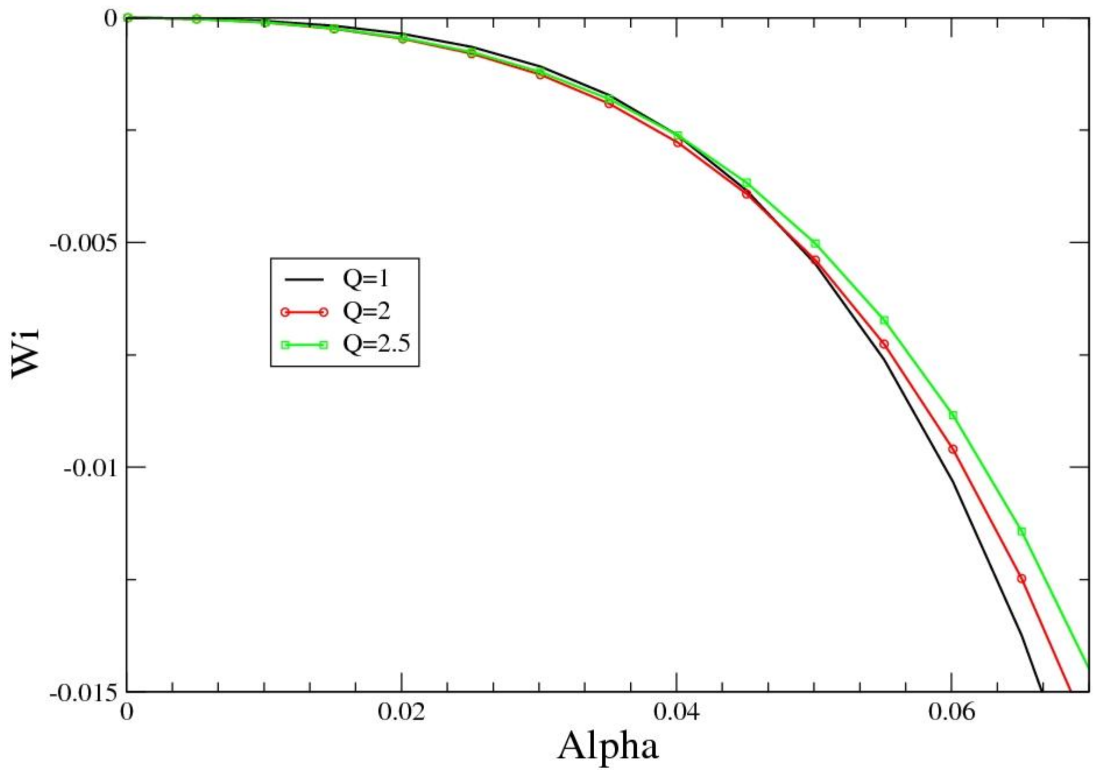

Fig. 14: Wi_{max} versus α for β = 20, Fr=0.4000 and various Q. (*branch 2*)

From the linear satiability analysis we can conclude that:

a) $Re_{critical}$ is a function of _ as well as Fr .

b) branch 1: decreasing Fr keeping Q and β constant has a stabilizing effect.

c) branch 1: increasing β keeping Fr and Q constant has a stabilizing effect.

d) branch 1: decreasing Q keeping Fr and β constant has a stabilizing effect.

e) branch 2: decreasing Fr keeping Q and β constant has a stabilizing effect.

f) branch 2: decreasing β keeping Fr and Q constant has a stabilizing effect.

g) branch 2: decreasing Q keeping Fr and β constant has a stabilizing effect.

5. Conclusions and Future Work

The effect of inlet conditions on base flow and its stability has been studied in detail. Depending on the inlet conditions flows may accelerate and become thinner or decelerate and experience a hydraulic jump along the incline. Base flow for various inlet conditions and inclinations has been obtained by numerically solving the weakly non-similar equations. The analysis here has been restricted to small inclination and small Reynolds Number flows where linear stability analysis is accurate. The linear stability of the base flow for various inlet conditions has been studied and neutral stability boundary has been compared to the expression obtained in for parabolic parallel flow down an incline by Yih, 1963.

This work can be extended to axis-symmetric flows like flow over a right circular cone. In the current work, nonlinear effects have been neglected. To study flow down inclination with a large physical angle accurately, nonlinear effects cannot be neglected. To understand flow driven instabilities in geo-physical formations like stalactites, stalagmites and icicles better the current work can be extended in 3-D and into the non-linear regime.

Appendix

MATLAB Code to iteratively solve the nonlinear eigenvalue problem for the most unstable mode:

%ose.m

```
clear all;
mu=1.8e-6;                          %kinematic viscosity
surfTe=69e-3;                       %surface tension
rho=1000;                           %density
N=40;                               %number of collocation points
tet=90;                             %angle of inclination
beta=tet*pi/180;
R0=[40,60,100];                     %Reynolds Number
for Re=1:3
    R=R0(Re);
    h=(2*R*(mu^2)/(9.8*sin(beta)))^(1/3);    %film thickness
    Uave=9.8*(h^2)*sin(beta)/(2*mu);          %mean velocity at surface
    F=Uave/(9.8*h)^0.5;                       %Froude's Number
    S=surfTe/(rho*h*(Uave^2));                %Non-dimensional surface tension
    [D,x]=cheb(N);                            %calling cheb.m (Defined in [Error! Reference source not found.])
    y=0.5*(1-x);
    D1=-2.*D;
    D=D1;
    U=(1-y.^2);                               %Mean flow
    D2=D^2;
    D3=D^3;
    D4=D^4;
    I=eye(N+1);
    count=1;
    cmin=2;                                   %c guess
```

```
for a=0.001:0.01:0.3
    error1=1;
    maxcnt=1;
    while error1>1e-6  && maxcnt<500
        A=(D4-2*(a^2)*D2+(a^4).*I) +1i*a*R*(-2).*I - 1i*a*R*diag(U)*(D2-(a^2).*I);
        B=-1i*a*R*(D2-(a^2).*I);
        TopB1=D2+(a^2).*I;
        B(1,:)=TopB1(1,:);                  %zero tangential stress
        B(N+1,:)=0;                         %no slip
        B(N,:)= 0;                          %no slip
        TopB2=-(a*R).*D;
        B(2,:)=TopB2(1,:);                  %normal stress balance

        A(N+1,:)=0;
        TopA1=(D2+(a^2+2).*I);
        A(1,:)=TopA1(1,:);                  %zero tangential stress
        A(N+1,N+1)=1;                       %no slip
        A(N,:)=D(N+1,:);                    %no slip
        TopA2=(a*(-R + 3i*a).*D -1i.*D3)+(1/(cmin-1))*(a*(2*cot(beta)+(a^2)*S*R)).*I  ;
        A(2,:)=TopA2(1,:);                  %normal stress balance
        ee=eig(A,B);
        cnt1=1;
        for z=1:N+1                         %loop to find the most unstable mode
            if abs(ee(z))<3
                t2(cnt1)=ee(z);
                cnt1=cnt1+1;
             end
        end
        for z=1:length(t2)
            if imag(t2(z))==max(imag(t2))
                cmin2=t2(z);
            end
```

```
            end
            error1=abs((cmin)-(cmin2));
            cmin=cmin2;                              %new c guess
            maxcnt=maxcnt+1;
        end
        c(count)=cmin;
        cre(count)=a*real(c(count));             %growth rate (Wr)
        cim(count)=a*imag(c(count));             %growth rate (Wi)
        count=count+1
    end
        alpt=[0.001:0.01:0.3];
        for z=1:length(alpt)        %Searching for the wave number of most unstable mode.
            if cim(z)==max(cim)
            amax=alpt(z);
        end
    end
    if Re==1                                     %Plotting
        figure, plot(alpt,cim,'-b.');
        lamb(1)=2*pi*h/amax;                     %wavelength of the most unstable mode
    elseif Re==2
        hold on, plot(alpt,cim,'-go');
        lamb(2)=2*pi*h/amax;
    else
         hold on, plot(alpt,cim,'-r+');
        lamb(3)=2*pi*h/amax;
    end
end
h = legend('Re=40','Re=60','Re=100',3);
set(h,'Interpreter','none')
lamb
```

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