Speleothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   Orr-Sommerfeld Equation   Conclusions   References

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Instabilities in Interfacial Flows: with Specific Reference to Stalactites and Stalagmites

Sarath Ramadurgam

November 12, 2009

Contents

Speleothems

Figure: Photo by Dave Bunnell showing the most common speleothems

A speleothem (from the Greek for "cave deposit"), commonly known as a cave formation, is a secondary mineral deposit formed in a cave. [1]

[1] *Source: wikipedia*

Speleothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   Orr-Sommerfeld Equation   Conclusions   References

Stalactites - A Model by M. B. Short et. al., 2005.

Key Parameters in Limestone Formations

- $CO_2 + H_2O \xrightarrow{k_{\pm 1}} H^+ + HCO_3^-$

- $CO_2 + OH^- \xrightarrow{k_{\pm 2}} HCO_3^-$

- $Ca^{2+} + HCO_3^- + OH^- \rightarrow CaCO_3 + H_2O$
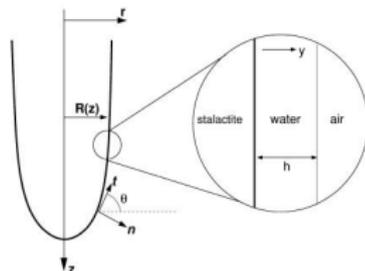
- $H^+ + CO_3^{2-} \rightarrow HCO_3^-$

Figure: Geometry of the surface of a stalactite [Short et. al., 2005.]

Parameter	Symbol	Value
Length	L	$10 - 100 cm$
Radius	R	$5 - 10 cm$
Film Thickness	h	$10 \mu m$
Fluid Velocity	u_c	$1 - 10 mm/s$
Reynolds Number	Re	$0.01 - 1.0$
Growth Rate	g_r	$1 cm/century$
Diffusion Time	τ_d	$0.1s$
Traversal Time	τ_t	$100s$
Growth Time	τ_g	$10^6 s$

- $y \equiv \frac{y}{h}$, $u_c \equiv \frac{gh^2 \sin(\theta)}{2\nu}$

- $u(y) = u_c(2y - y^2)$

- $\tau_d = \frac{h^2}{D}$, $\tau_t = \frac{L}{u_c}$ and $\tau_g = \frac{h}{g_r}$

- $Re = \frac{u_c h}{\nu}$

Speleothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   Orr-Sommerfeld Equation   Conclusions   References
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Stalactites - A Model by M. B. Short et. al., 2005.

Reaction-Diffusion Equation and Growth Rate

-

$$x \equiv \frac{x}{L}, \ u \equiv \frac{u}{u_c}, \ C = C_0(1 + \phi), \ \delta = \sqrt{\frac{h^2 k_+}{D}}, \ w = \frac{k_-[HCO_3^-]}{k_+[CO_2]_0} - 1, \ \epsilon = \frac{h}{R}$$

-

$$R_{CO_2} = k_-[HCO_3^-] - k_+[CO_2], \ k_+ = k_{+1} + k_{+2}[OH^-], \ k_- = k_{-1}[H^+] + k_{-2}$$

-

$$\frac{\tau_d}{\tau_t} u \frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} + (\frac{h}{L})^2 \frac{\partial^2 \phi}{\partial x^2} + \delta(w - \phi)$$

-

$$\phi = w\delta^2 [\frac{1 - y^2}{2} - \epsilon \frac{1 - y^3}{3} + \frac{DH}{D_a} \frac{1 - \epsilon - \epsilon^2}{\epsilon}]$$

-

$$F \propto h, \ h = (\frac{3Q\nu}{2\pi g R \sin(\theta)})^{\frac{1}{3}}$$

-

$$\widehat{n}.v \propto [\frac{1}{r \sin(\theta)}]^{\frac{1}{3}}$$

Stalactites and Icicles

Assumptions in Stalactite Model:

- The flow is low Reynolds, laminar and steady.
- Film thickness is very small, hence 2-D parallel flow approximation.
- Non-liner terms in the Reaction-Diffusion equation are been dropped.

Similarities with Icicles:

- Icicles and stalactites grow into conical structures.
- Both show surface ripples of the order of 6-10mm.
- There are many similarities in the physics of formation.

Spelothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   Orr-Sommerfeld Equation   Conclusions   References

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Flow Along an Incline

Governing Equations

Linearized Navier-Stokes Equations:

- $x \equiv \frac{x}{h}$, $p \equiv \frac{p}{\rho u_a^2}$, $t = \frac{t u_a}{h}$, $u_a = \frac{g h^2 \sin(\beta)}{2\nu}$, $Re = \frac{u_a h}{\nu}$, $F = \frac{u_a}{\sqrt{gh}}$, $S = \frac{T}{\rho h u_a^2}$

- U is the mean flow and u, v, p are sinusoidal perturbations.

- $u_t + U u_x + U_y v = -p_x + (\frac{1}{Re})\Delta u$

- $v_t + U v_x = -p_y + (\frac{1}{Re})\Delta v$

- $u_x + v_y = 0$

- $u = \psi_y$ and $v = \psi_x$

Boundary Conditions:

- $\psi_y = 0$, $-\psi_x = 0$ at $y = \xi$ (no slip)

- $U_{yy}\eta + \psi_{yy} - \psi_{xx} = 0$ at $y = 1$ (zero tangential stress)

- $[\frac{\cos(\beta)}{F^2}]\eta + p + [\frac{1}{Re}]\psi_{xy} - S\eta_{xx}$ at $y = 1$ (normal stress balance)

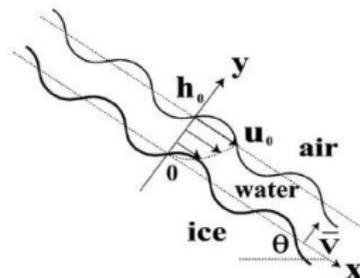- $-\psi_x = \eta_t + U\eta_x$ at $y = 0$ (kinematic condition)

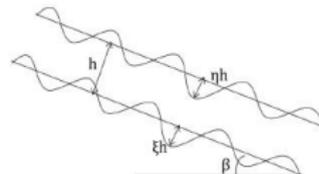Figure: Geometry of the surface of an Icicle [Ueno, 2007.]

Figure: Instability in Fluid Flow Along an Incline

Stream Function

Assumptions in Ueno's Icicle model:

- $\xi = \xi_k \exp[\sigma t - ikx]$ where $k = \frac{2\pi}{\lambda}$ (not scaled), $\alpha = kh$
- $\psi(y) = f(y)\xi_k$ (small surface corrugations)
- Long wavelength ($\alpha^2 \cong 0$) and quasistationary ($\sigma = 0$) approximation

On substituting the above assumptions, the Orr-Sommerfeld equation and the boundary conditions reduce to:

- $\frac{d^4 f}{dy^4} = i\alpha Re\{U\frac{d^2 f}{dy^2} - \frac{d^2 U}{dy^2}f\}$
- $f\xi_k = -\eta_k$ at $y = 0$ (kinematic condition)
- $f_y\xi_k + U_y\xi_k = 0$ and $f\xi_k = 0$ at $y = 0$ (no slip)
- $f_{yy}\xi_k + U_{yy}\eta = 0$ at $y = 1$ (zero tangential stress)
- $f_{yyy}\xi_k - i\alpha Re\{\frac{\cos(\beta)}{F^2} + \alpha^2 S\}\eta_k = 0$ at $y = 1$ (normal stress balance)
- αRe is very small, thus solving for $\frac{d^4 f}{dy^4} = 0$
- $f(y) = -2y + \frac{3(2-iq)}{6-iq}y^2 + \frac{ia}{6-iq}y^3$ where $q = \alpha Re\{\frac{\cos(\beta)}{F^2} + \alpha^2 S\}$

Speleothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   Orr-Sommerfeld Equation   Conclusions   References

Heat Flow in an Icicle

Governing Equations

- T_l, T_s, T_a and g_l, g_s, g_a are the mean temperatures and temperature perturbations in liquid, solid and air respectively.
- $\frac{\partial T_l}{\partial t} + u \frac{\partial T_l}{\partial x} + v \frac{\partial T_l}{\partial y} = \kappa_l \Delta T_l$ where κ is the thermal diffusivity.
- $\frac{\partial T_s}{\partial t} - \overline{V} \frac{\partial T_s}{\partial y} = \kappa_s \Delta T_s$ where $\overline{V}$ is a reference frame moving at the solid-liquid interface
- $\frac{\partial T_a}{\partial t} - \overline{V} \frac{\partial T_a}{\partial y} = \kappa_a \Delta T_a$
- Linear mean temperature profiles assumed.
- Continuity of temperature at solid-liquid and liquid-air interface and heat conservation at the liquid-air interface is applied to solve the equations.
- Temperature at liquid-air interface is considered constant. Thus the perturbation temperature at $y = \eta$ is identically zero for both air and liquid.

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Growth Rate

Dispersion Relation

- Heat conservation at solid-liquid interface: $L\{\overline{V} + \frac{\partial \xi}{\partial t}\} = K_s \frac{\partial T_s}{\partial y} - K_l \frac{\partial T_l}{\partial y}$ at $y = \xi$, where K is the thermal conductivity.

- On linearizing the above equation: $L\sigma \xi_k = K_s \frac{\partial g_s}{\partial y} - K_l \frac{\partial g_l}{\partial y}$

Figure: Dispersion Relation [Ueno, 2007.]

Speleothems   Icicles - A Model by K. Ueno, 2007   **Modeling Ripples on Stalactites**   Orr-Sommerfeld Equation   Conclusions   References

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Governing Equations, Assumptions and Growth Law

Stalactites and Icicles (Revisited)

- The Heat flow equation is similar to the reaction-diffusion equation.
- Mean concetration profile unlike the temperature cannot be assumed to be linear.
- Under long wavelength and quasistationary assumptions, the solution to Orr-Sommerfeld equation is the same as that for Icicles.
- The mean concentration profile for 2-D flow is calculated by assuming a diffusion boundary layer (ζ) of CO_2 in air and Henry's Law.
- $\overline{C} = C_0(1 + \phi)$, for 2-D $\phi = w\delta^2\{\frac{1-y^2}{2} + \zeta\frac{DH}{D_a}\}$ where
  $w = \frac{k_-[HCO_3^-]}{k_+[CO_2]_0} - 1, \ \delta = \sqrt{\frac{h^2 k_+}{D}}$
- $\zeta \equiv \frac{1}{\epsilon}(= \frac{R}{h})$
- The perturbation of concentration in liquid (g) has a form similar to Icicles (confluent hypergeometric function with reaction term changing the value of the constant in $_1F_1$)
- Henry's Law and conservation of mass are applied at liquid-air interface. No penetration condition is applied at the solid-liquid interface.

Speleothems   Icicles - A Model by K. Ueno, 2007   **Modeling Ripples on Stalactites**   Orr-Sommerfeld Equation   Conclusions   References

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Governing Equations, Assumptions and Growth Law

Growth Law

- The growth law for stalactite is obtained by equating the flux of CO_2 diffusion through the liquid surface to the flux of $CaCO_3$ deposition at the solid surface.

- $\frac{\partial \xi}{\partial t} \propto -D \frac{\partial g}{\partial y}$ at $y = \eta$

- $-i\alpha\gamma\xi_k \propto -D\frac{\partial g}{\partial y}$ at $y = 1$

- $\gamma = \frac{iw\delta^2}{\alpha}\{B + i\alpha Pef_0\}$ where Pe is the Peclet number, $f_0 = f(0)$

- $B =$
$\frac{\alpha D_a + f_0(-\alpha D_a + (1-\alpha)DH)\{\delta^2 + \frac{2i\alpha Pe}{3} - \frac{\alpha^2 Pe^2}{20}\}}{DH\{\delta^2 + \frac{2i\alpha Pe}{3} - \frac{\alpha^2 Pe^2}{20}\} + \alpha D_a\{1 + \frac{\delta^2}{2} + \frac{i\alpha Pe}{2} - \frac{\alpha^2 Pe^2}{180}\}}$

- $Re = 1$ gives $\lambda = 3.3mm$ and $Re = 0.01$ gives $\lambda = 2.4mm$. [a]

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[a] $\beta = \frac{\pi}{2}$, $\nu = 1.8 \times 10^{-6} m^2 s^{-1}$, $T = 69 \times 10^{-3} Nm^{-1}$, $\rho = 1000 Kgm^{-3}$

Figure: Form of the Dispersion Relation for Stalactites

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Incompleteness of the Model

- Quasistationary assumption may not be valid.
- Circular effects may be significant.
- Corrugation amplitude is not small.
- Flow may not be parallel due to large corrugations.
- Ueno assumes the perturbation values of temperature to go to zero at liquid surface. Though this assumption has not been made for stalactites, Henry's Law is assumed to hold for the perturbation.
- The above simplification holds when high powers of αPe are small compared to the coefficients in the confluent hyper-geometric function. This may not be very accurate for cases with $Re > 0.5$ where $Pe > 900$.

Non-Linear Eqigenvalue Problem

Linearized Navier-Stokes Equations:

- $u_t + U u_x + U_y v = -p_x + (\frac{1}{Re})\Delta u$
- $v_t + U v_x = -p_y + (\frac{1}{Re})\Delta v$
- $u_x + v_y = 0$
- $u = \psi_y$, $v = \psi_x$
- $\psi(y) = \phi(y)e^{i\alpha(x-ct)}$ and $p(y) = \vartheta(y)e^{i\alpha(x-ct)}$

Orr-Sommerfeld Equation:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha Re\{(U-c)(\phi'' - \alpha^2\phi) - U''\phi\}$$

Boundary Conditions (Assuming no Corrugations):

- $\eta_k = \frac{\phi(1)}{c - U(1)}$ (kinematic condition)
- $\phi'(0) = 0$, $\phi(0) = 0$ (no slip)
- $\phi''(1) + (\alpha^2 - \frac{U''(1)}{c - U(1)})\phi(1) = 0$ (zero tangential stress)
- $\{\frac{\alpha Re}{F^2} + \alpha^3 SRe\}\frac{\phi(1)}{c - U(1)} + \alpha Re U'(1)\phi(1) + \{(c - U(1))\alpha Re + 3i\alpha^2\}\phi'(1) - i\phi'''(1) = 0$ (normal stress balance)

Numerical Results

$\{\frac{\cos(\beta)}{F^2} + \alpha^3 SRe\}\frac{\phi(1)}{c-U(1)} + \{(c - U(1))\alpha Re + 3i\alpha^2\}\phi'(1) - i\phi'''(1)$

The numerical results for the most unstable mode using Chebyshev Differential matrices: [2]
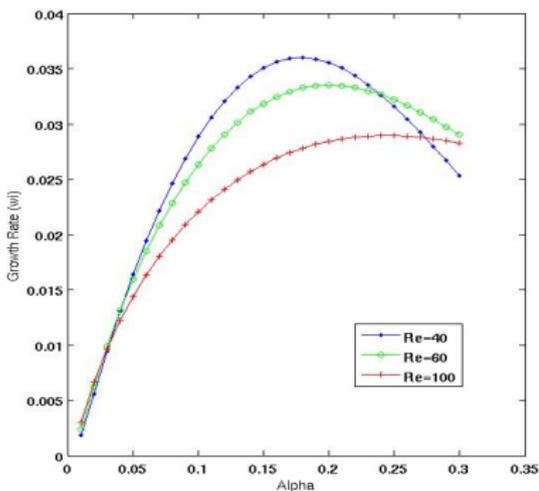

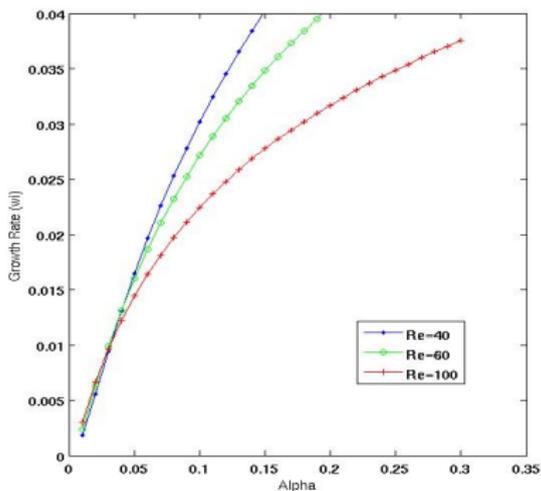Figure: Dispersion Relation (ω_i vs. α) with non-zero Surface Tension

Figure: Dispersion Relation (ω_i vs. α) with zero Surface Tension

[2] $\beta = \frac{\pi}{2}$, $\nu = 1.8 \times 10^{-6} m^2 s^{-1}$, $T = 69 \times 10^{-3} Nm^{-1}$, $\rho = 1000 Kgm^{-3}$

Speleothems Icicles - A Model by K. Ueno, 2007 Modeling Ripples on Stalactites **Orr-Sommerfeld Equation** Conclusions References

Flow Along a Cylinder

Non-Linear Eqigenvalue Problem

Orr-Sommerfeld Equation:

$$[(c - U)(D^2 - \alpha^2) + D^2 U]\phi = \frac{i}{\alpha Re}(D^2 - \alpha^2)^2 \phi$$

$$D = r\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}), \ \gamma = \frac{r_0}{h}, U(r) = (2r - r^2)$$

Boundary Conditions (Assuming no Corrugations):

- $\phi'(\gamma) = 0$, $\phi(\gamma) = 0$ (no slip)
- $\phi''(\gamma + 1) - \frac{1}{r}\phi'(\gamma + 1) + (\alpha^2 - \frac{R}{F^2(c-1)})\phi(\gamma + 1) = 0$ (zero tangential stress)
- $\{-(\alpha^2 - \frac{1}{r^2})\alpha SRe\}\frac{\phi(\gamma+1)}{c-1} - \frac{2i\alpha^2}{r}\phi(\gamma + 1) + \{(c - 1)\alpha Re + i(3\alpha^2 - \frac{1}{r^2})\}\phi'(\gamma + 1) + \frac{i}{r}\phi''(\gamma + 1) - i\phi'''(\gamma + 1) = 0$ (normal stress balance)
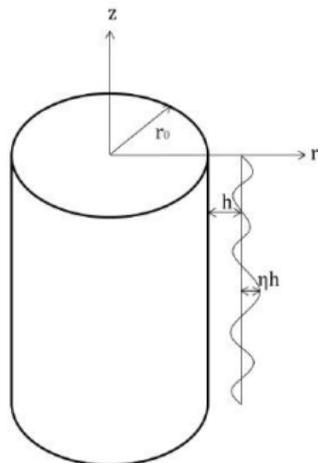

Figure: Flow along a cylinder

Numerical Results

The numerical results for the most unstable mode using Chebyshev Differential matrices: [3]

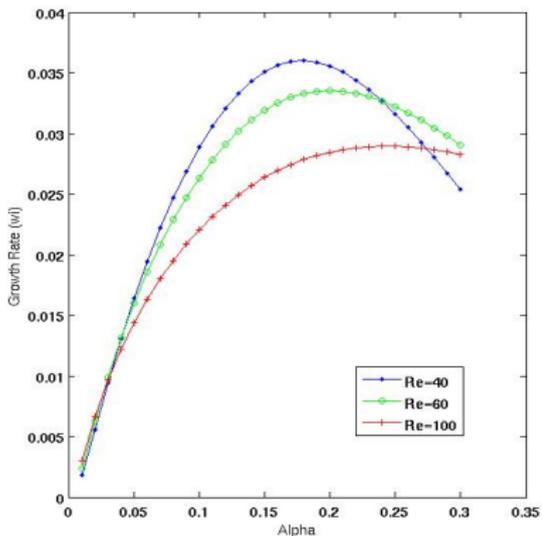Figure: Dispersion Relation (ω_i vs. α) for $r_0 = 0.1m$
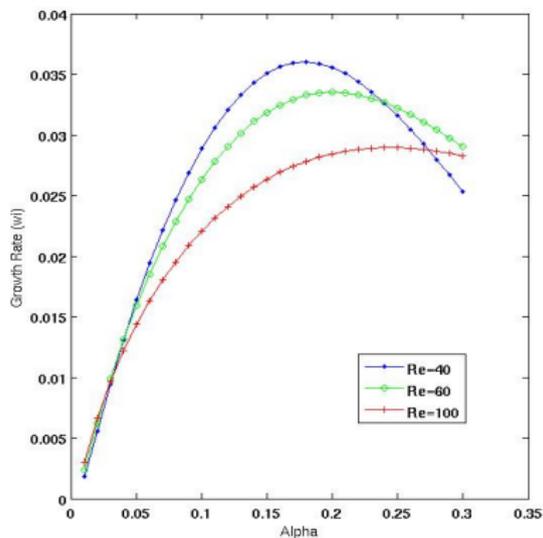

Figure: Dispersion Relation (ω_i vs. α) for 2-D incline flow

[3] $\beta = \pi/2$, $\nu = 1.8 \times 10^{-6} m^2 s^{-1}$, $T = 69 \times 10^{-3} Nm^{-1}$, $\rho = 1000 Kgm^{-3}$

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Flow Along a Cylinder

Numerical Results

Re	h	$\gamma(r_0 = 2mm)$	$\gamma(r_0 = 1mm)$
40	297.95 μm	6.7	3.3
60	341.37 μm	5.8	2.9
100	404.37 μm	4.9	2.4
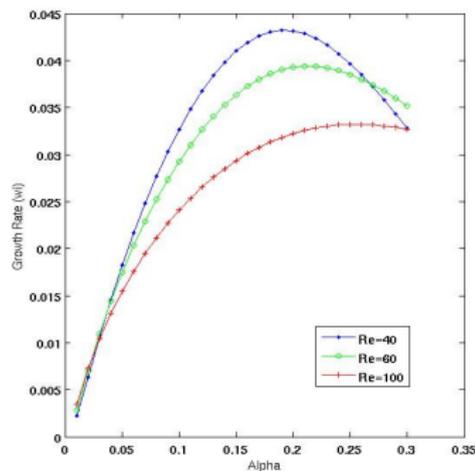
Figure: Dispersion Relation (ω_i vs. α) for $r_0 = 0.002m$

Figure: Dispersion Relation (ω_i vs. α) for $r_0 = 0.001m$

Speleothems   Icicles - A Model by K. Ueno, 2007   Modeling Ripples on Stalactites   **Orr-Sommerfeld Equation**   Conclusions   References

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Small Corrugations

Small Corrugations in 2-D Flow

Orr-Sommerfeld Equation:

$$\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi = i\alpha Re\{(U - c)(\phi'' - \alpha^2 \phi) - U''\phi\}$$

Boundary Conditions (Assuming small Corrugations):

- $\eta_k = \frac{\phi(1)}{c - U(1)}$ (kinematic condition)
- $\phi'(\xi) = 0$, $\phi(\xi) = 0$ (no slip)
- $\phi'(0) + U'(0)\xi_k = 0$, $\phi(0) = 0$ (no slip) [a]
- $\phi''(1) + (\alpha^2 - \frac{U''(1)}{c - U(1)})\phi(1) = 0$ (zero tangential stress)
- $\{\frac{\alpha Re}{F^2} + \alpha^3 SRe\}\frac{\phi(1)}{c - U(1)} + \alpha ReU'(1)\phi(1) +$ $\{(c - U(1))\alpha Re + 3i\alpha^2\}\phi'(1) - i\phi'''(1) = 0$ (normal stress balance)
- If $\psi = \phi(y)e^{i\alpha(x-ct)} = \chi(y)\xi_k e^{i\alpha(x-ct)}$, then the no slip condition becomes $\chi'(0) + U'(0) = 0$ which is an inhomogenous boundary condition.

1.1cm

1.3cm

Figure: Icicle [Ueno, 2007]
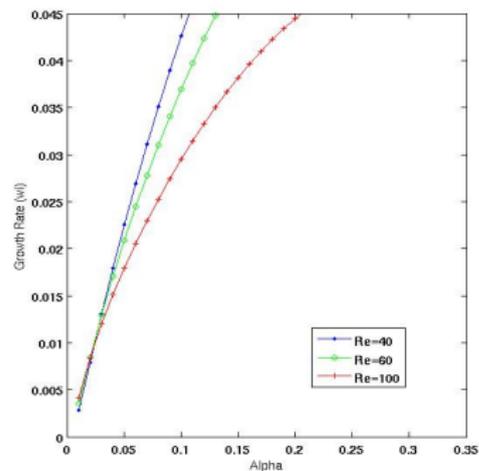
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[a] Using the boundary condition in this form does not cause a sensible change in the dispersion relation

Future Work

- The quantitative effect of quasi-stationary assumption has to be understood
- 2-D flow assumption is valid for $r_0 > 10h(approx.)$
- The effect of small corrugations on the incline for a flow has not been completely studied. Wierschem et al., 2008 study this case for perturbations of the form $\psi = \phi e^{ikx}$
- Amplitude of ripples in icicles is greater than the film thickness ($100\mu m$) thus parallel flow assumption may not be valid.
- The effect of disjoining pressure might be of importance in the case of stalactites.

<div align="center">Thank You.</div>

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